On weighted means, geometric and arithmetic

Declaration

The integer constant N satisfies $1 \le N$.

The integer dummy i is restricted to the range $0 \le i < N$.

The function p, with $p \in \mathbb{N} \to \mathbb{R}$, satisfies

(0) $\langle \forall i :: 0 \le p.i \rangle \wedge \langle \Sigma i :: p.i \rangle = 1$.

For any function q , with $q \in \mathbb{N} \to \mathbb{R}$, the weighted average $w.q$ is defined by

(1) $w.q = \langle \Sigma i :: p.i * q.i \rangle$.

[If we so desire, we can recognize a scalar product.]

Finally, function a , with $a \in \mathbb{N} \to \mathbb{R}$, satisfies

$$\langle \forall i :: 0 \le a.i \rangle \; .$$

* * *

The well-known fact that the geometric mean is at most the arithmetic mean, i.e.,

(2) $\sqrt[N]{\langle \Pi i :: a.i \rangle} \le \langle \Sigma i :: a.i \rangle / N$

is a special case of

(3) $\langle \Pi i :: a.i^{p.i} \rangle \le \langle \Sigma i :: p.i * a.i \rangle$

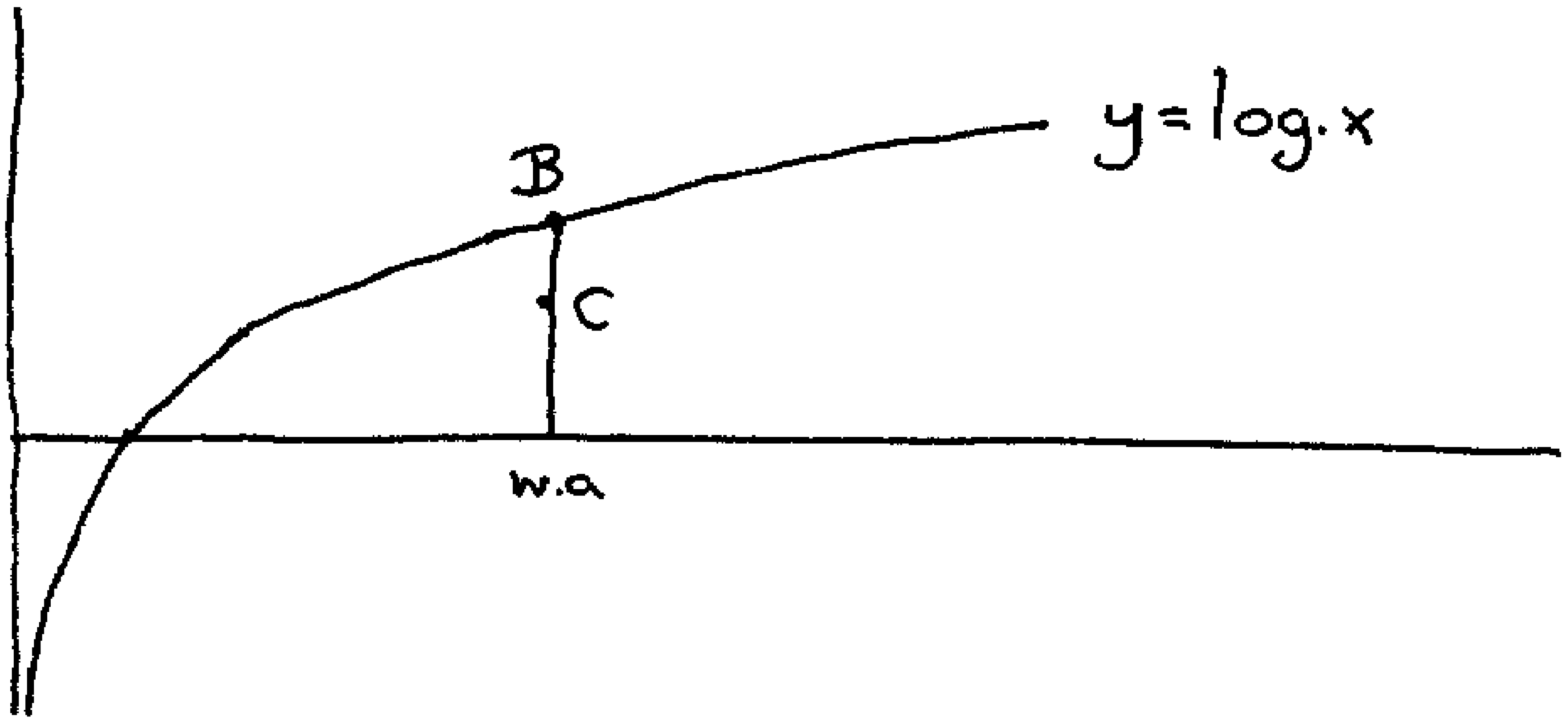
[Take $p.i = 1/N$, and (3) reduces to (2).] Our target is to prove (3). Taking logarithms at both sides, we are led to consider the relation

(4) $\quad w.(\log \circ a) \leq \log.(w.a)$,

where $\circ$ denotes functional composition. Because of the monotonicity of the exponential function, (4) implies (3), and our target is now to prove (4) .

Consider in the plane the weights p.i placed at Q.i , with Q.i = (a.i, log.(a.i)). Their centre of gravity C is by definition and (1)

(5) $\quad C = (w.a,\ w.(\log \circ a))$.



We now use the fact for p satisfying (0) that with weights p.i placed on a convex body, their centre of gravity does not lie outside the convex body (in general it lies inside). [Note that this is a possible definition of convexity.]

Since the second derivative of log.x is negative, the curve y = log.x is convex, and hence the centre of gravity C of non-negative weights placed on the curve lies on or under the curve, more precisely on or below point B given by

$$(6) \qquad B = (w.a\ ,\ \log.(w.a))\ .$$

Comparing their y-coordinates - see (5) and (6), we observe

$$w.(\log^\circ a) \le \log.(w.a)$$

which had to be proved.

Acknowledgement Thanks to the members of the ATAC that attended yesterday's session where this probleem was discussed. The above proof I designed last night. I found the theorem (3) in [0, p 102] (and prefer my proof).

[0] Aigner, Martin & Ziegler, Günter M., "Proofs from THE BOOK", Springer-Verlag Berlin, Heidelberg, New York, 1998, [ISBN 3-540-63698-6]

Austin, 9 February 2000

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