Beware of the empty range

Recently David Gries drew my attention to the problem of proving

(0) $\langle \exists x :: \langle \forall y :: p.x \equiv p.y \rangle\rangle \equiv \langle \exists x :: p.x \rangle \equiv \langle \forall y :: p.y \rangle$,

in which all quantifications are understood to be over the same range. As soon as I had seen the problem, I put David's text away and tackled the problem myself. In doing so I had a surprise; hence this little note.

* * *

My surprise was, that I had to introduce a case analysis, as soon as I had decided to prove (0) by demonstrating (1) and (2):

(1) $\langle \exists x :: \langle \forall y :: p.x \equiv p.y \rangle\rangle \equiv \langle \forall x,y :: p.x \equiv p.y \rangle$

(2) $\langle \exists x :: p.x \rangle \equiv \langle \forall y :: p.y \rangle \equiv \langle \forall x,y :: p.x \equiv p.y \rangle$.

While, because $false \equiv false \equiv true$, (0) definitely holds for empty ranges, (1) and (2) are false for empty ranges! So now we know that in the proofs of (1) and (2) the non-emptiness of the ranges has to be used. We use (half of) the

Lemma. For any q and quantifications over the same range

(3) $\langle \exists z :: q.z \rangle \Leftarrow \langle \forall z :: q.z \rangle$ for non-empty range

(4) $\langle \exists z :: q.z \rangle \Rightarrow \langle \forall z :: q.z \rangle$ for empty range

Proof of (3) We observe for any q and non-empty range

$$
\begin{array}{ll}
 & \langle \forall z :: q.z \rangle \Rightarrow \langle \exists z :: q.z \rangle \\
\equiv & \quad \{\text{predicate calculus; de Morgan}\} \\
 & \langle \exists z :: \neg q.z \rangle \vee \langle \exists z :: q.z \rangle \\
\equiv & \quad \{\text{combining the terms}\} \\
 & \langle \exists z :: \neg q.z \vee q.z \rangle \\
\equiv & \quad \{\text{excluded middle}\} \\
 & \langle \exists z :: true \rangle \\
\equiv & \quad \{\text{range non-empty}\} \\
 & true .
\end{array}
$$

(End of Proof of (3).)

Proof of (1) We observe for any p and non-empty ranges

$$
\begin{array}{ll}
 & \langle \exists x :: \langle \forall y :: p.x \equiv p.y \rangle\rangle \\
\equiv & \quad \{\wedge \text{ idempotent ; renaming the dummies}\} \\
 & \langle \exists z :: \langle \forall y :: p.z \equiv p.y \rangle \wedge \langle \forall x :: p.z \equiv p.x \rangle\rangle \\
\equiv & \quad \{\text{ranges } x, y, z \text{ the same}\} \\
 & \langle \exists z :: \langle \forall x, y :: (p.z \equiv p.y) \wedge (p.z \equiv p.x) \rangle\rangle \\
\Rightarrow & \quad \{\text{Leibniz; quantifications are monotonic}\} \\
 & \langle \exists z :: \langle \forall x, y :: p.x \equiv p.y \rangle\rangle \\
\Rightarrow & \quad \{\text{quantified constant}\} \\
 & \langle \forall x, y :: p.x \equiv p.y \rangle \qquad (*) \\
\equiv & \quad \{\text{unnesting}\} \\
 & \langle \forall x :: \langle \forall y :: p.x \equiv p.y \rangle
\end{array}
$$

$$\Rightarrow \quad \{\text{range } x \text{ non-empty; } (3)\}$$
$$\langle \exists x :: \langle \forall y :: p.x \equiv p.y \rangle \rangle$$

First line, the line marked (*) and the last line prove for non-empty ranges (1) by mutual implication.

(End of Proof of (1).)

<u>Proof of (2)</u> We observe for any p and non-empty ranges

$$\langle \exists x :: p.x \rangle \equiv \langle \forall y :: p.y \rangle$$
$$\equiv \quad \{\text{mutual implication; range non-empty, } (3)\}$$
$$\langle \exists x :: p.x \rangle \Rightarrow \langle \forall y :: p.y \rangle$$
$$\equiv \quad \{\text{predicate calculus}\}$$
$$\langle \forall x, y :: p.x \Rightarrow p.y \rangle$$
$$\equiv \quad \{\wedge \text{ idempotent; renaming the dummies}\}$$
$$\langle \forall x, y :: p.x \Rightarrow p.y \rangle \wedge \langle \forall x, y :: p.y \Rightarrow p.x \rangle$$
$$\equiv \quad \{\text{combining the terms; mutual implication}\}$$
$$\langle \forall x, y :: p.x \equiv p.y \rangle$$

and thus (2) has been proved for non-empty ranges.

(End of Proof of (2).)

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prof. dr Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA