# Sylvester's theorem used (see EWD1016)

On Tuesday 2 January 1996, Ronald W. Bulterman told me the following theorem.

Consider, for $n \geq 2$, $n$ distinct points in the plane; let $k$ be the number of all distinct lines such that each of them contains at least 2 of those points. Then $k = 1 \lor k \geq n$ .

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Proof The proof is by mathematical induction on $n$ . There are two reasons for trying this proof structure, firstly the shape of the disjunct $k \geq n$ in the demonstrandum and secondly the fact that (Euclid's Axiom) $n=2 \Rightarrow k=1$ immediately establishes the base.

With $n' = n+1$ , we consider for the induction step $n'$ distinct points, giving rise to $k'$ lines. We have to show $k'=1 \lor k' \geq n'$ , where we may use $k=1 \lor k \geq n$ "ex hypothese". In view of our demonstrandum being a disjunction, we introduce a case analysis, viz. $k'=1$ versus $k' \neq 1$ .

$k' = 1$

In this case, the demonstrandum $k'=1 \vee k' \geq n'$ follows directly (i.e. by predicate calculus alone).

$k' \neq 1$

In this case, the demonstrandum $k'=1 \vee k' \geq n'$ simplifies directly to $k' \geq n'$. The remainder of the proof is devoted to showing how, for $n'$ noncollinear points, $k' \geq n'$ follows from the induction hypothesis.

In order to be able to appeal to the induction hypothesis, we single out one of the $n'$ points —let us call that point "A"— and consider the remaining $n$ points and the $k$ lines they give rise to all by themselves. Ex hypothese we may use $k=1 \vee k \geq n$ ; we exploit the two disjuncts separately.

In the case $k=1$, the $n$ (distinct) points lie on a single line, and, because the $n'$ points are noncollinear, that line does <u>not</u> contain $A$. Hence $k' = n+1$, and since $n' = n+1$, $k' \geq n'$ has been established.

In the remaining case $k \geq n$ —or, since $n' = n+1$, equivalently $k+1 \geq n'$—, our demonstrandum $k' \geq n'$ follows from $k' \geq k+1$ or, equivalently, $k' > k$. How do we establish $k' > k$? Or, in other words, how can we conclude that the removal of A reduces the number of lines? Well, since a line has to go through at least 2 points of the set, such a line disappears if A is one of the only 2 points it goes through. Hence we can assert $k' > k$ by a proper choice of A provided:

"For any number of distinct, noncollinear points in the plane, there exists a line through exactly 2 of them."

But this was Sylvester's conjecture, which since then became a theorem, so we are done.

(End of Proof.)

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